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Extension of Matrix Algebra and Linear spaces of Linear Transformations

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Abstract: **Here we define 'addition' and 'multiplication' on the set *M(F)* of all matrices over a field *F* as an extension of traditional matrix addition and multiplication respectively and study about the algebraic structure $(\boldsymbol{M}(\boldsymbol{F}),+,.)$.**

Again, since a matrix can be thought as a linear transformation from a vector space to a vector space over a given field *F*, we shall have a kind of extension of all linear spaces of linear transformations over the field *F*.

Keywords: **addition of matrices, extension of linear spaces, extension of matrix algebra, multiplication of matrices, weak hemi-ring, weak hemi-vector space.**

NOTATIONS: (i) $M_{m\times n}(F)$ denotes the set of all $m\times n$ matrices over a given field F, for some positive integers m, n.

(ii) $A_{m\times n}\in M(F)$ denotes $A_{m\times n}$ is an $m\times n$ matrix in $M(F)$, for some positive integers m, n.

(iii) $M_n(F)$ denotes the set of all $n\times n$ matrices over a given field F.

(iv) $M_{[n]}(F)$ is the set of all matrices in $M(F)$ each having n columns, for some given positive integer n.

(v) $M_{(n)}(F)$ is the set of all matrices in $M(F)$ each having n rows, for some given positive integer n.

(vi) $O_{m\times n}$ denotes the $m\times n$ matrix in $M(F)$, of which all the elements are zero.

(vii) If $A_{m\times n}=\left(a_{ij}\right)_{m\times n}\in M(F)$ and p, q are positive integers such that $p\leq m,\ q\leq n$, then $A_{p\times q}=\left(a_{ij}\right)_{p\times q}$.

1. INTRODUCTION

In traditional matrix algebra, it is clear that 'addition' on $M_{m\times n}(F)$ and 'addition' on $M_{p\times q}(F)$ are not same binary operations, provided$(m,n)\neq(p,q)$.

Also traditional matrix multiplication is a mapping from $M_{[n]}(F)\times M_{(n)}(F)$ to $M(F)$ so that matrix multiplication is not a binary operation on a set of matrices. Also, for different values of n, we shall get different matrix multiplications. Therefore the statement "matrix multiplication is associative" is not meaningful.

Observing it we get motivation to extend matrix addition and multiplication as binary operations on the set *M(F)* and study about the algebraic structure $(M(F),+,.)$ following [1], [2], [3], [4], [5], [6], [7], [8]. And we get an extension of all linear spaces of linear transformations over the field F.

To define matrix addition in $M(F)$, firstly we embed the given matrices into matrices of suitable higher order, and then add.

2. MAIN RESULTS

➢ **Extension of Matrix Algebra :**

Definition (2.1) Define 'addition' of matrices in $M(F)$ by for all $A_{m\times n} = (a_{ij})_{m\times n}$, $B_{p\times q} = (b_{ij})_{p\times q} \in M(F)$,

$A_{m\times n} + B_{p\times q} = (c_{ij})_{r\times s}$, where $r = \max\{m,p\}$, $s = \max\{n,q\}$ and for $= 1,2,\ldots,r$; $j = 1,2,\ldots,s$, $c_{ij} = a'_{ij} + b'_{ij}$,

where $a'_{ij} = \begin{cases} a_{ij}, & if\ 1 \le i \le m,\ 1 \le j \le n \\ 0, & otherwise \end{cases}$, for $i = 1,2,\ldots,r$; $j = 1,2,\ldots,s$ and

$b'_{ij} = \begin{cases} b_{ij}, & if\ 1 \le i \le p,\ 1 \le j \le q \\ 0, & otherwise \end{cases}$, for $i = 1,2,\ldots,r$; $j = 1,2,\ldots,s$.

Example (2.1) Consider the real matrices

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & -1 & 3 \end{pmatrix}_{3\times 4}, \quad B = \begin{pmatrix} 9 & 1 \\ 2 & 8 \\ 7 & 3 \\ 4 & 6 \\ -1 & 4 \end{pmatrix}_{5\times 2}.$$

Then
$$A + B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{5\times 4} + \begin{pmatrix} 9 & 1 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 4 & 6 & 0 & 0 \\ -1 & 4 & 0 & 0 \end{pmatrix}_{5\times 4} = \begin{pmatrix} 10 & 3 & 3 & 4 \\ 7 & 14 & 7 & 8 \\ 16 & 3 & -1 & 3 \\ 4 & 6 & 0 & 0 \\ -1 & 4 & 0 & 0 \end{pmatrix}_{5\times 4}$$

Theorem (2.1) $(M(F),+)$ is a commutative monoid.

Proof : Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{p\times q}$, $C = (c_{ij})_{r\times s} \in M(F)$ be arbitrary.

Then $A + B = (a'_{ij})_{u_1\times v_1} + (b'_{ij})_{u_1\times v_1} = (a'_{ij} + b'_{ij})_{u_1\times v_1} = (d_{ij})_{u_1\times v_1}$,

where $u_1 = \max\{m,p\}$, $v_1 = \max\{n,q\}$ and

$a'_{ij} = \begin{cases} a_{ij}, & if\ 1 \le i \le m,\ 1 \le j \le n \\ 0, & otherwise \end{cases}$, for $i = 1,2,\ldots,u_1$; $j = 1,2,\ldots,v_1$,

$b'_{ij} = \begin{cases} b_{ij}, & if\ 1 \le i \le p,\ 1 \le j \le q \\ 0, & otherwise \end{cases}$, for $i = 1,2,\ldots,u_1$; $j = 1,2,\ldots,v_1$ and

$d_{ij} = a'_{ij} + b'_{ij}$, for $i = 1,2,\ldots,u_1$; $j = 1,2,\ldots,v_1$. .... (1)

Therefore, for $i = 1,2,\ldots,u_1$; $j = 1,2,\ldots,v_1$, $d_{ij} = a'_{ij} + b'_{ij} = b'_{ij} + a'_{ij}$

(since F is a field and $(a'_{ij})_{u_1\times v_1}$, $(b'_{ij})_{u_1\times v_1} \in M(F)$).

But $(b'_{ij} + a'_{ij})_{u_1\times v_1} = B + A$.

Therefore 'addition' is commutative .

Again $B + C = (b''_{ij})_{u_2\times v_2} + (c'_{ij})_{u_2\times v_2} = (b''_{ij} + c'_{ij})_{u_2\times v_2} = (e_{ij})_{u_2\times v_2}$, where $u_2 = \max\{p,r\}$, $v_2 = \max\{q,s\}$ and

$b''_{ij} = \begin{cases} b_{ij}, & if\ 1 \le i \le p,\ 1 \le j \le q \\ 0, & otherwise \end{cases}$, for $i = 1,2,\ldots,u_2$; $j = 1,2,\ldots,v_2$,

$c'_{ij} = \begin{cases} c_{ij}, & if\ 1 \le i \le r,\ 1 \le j \le s \\ 0, & otherwise \end{cases}$, for $i = 1,2,\ldots,u_2$; $j = 1,2,\ldots,v_2$ and

$e_{ij} = b''_{ij} + c'_{ij}$, for $i = 1,2,\ldots,u_2$; $j = 1,2,\ldots,v_2$ . ... (2)

Now $(A+B)+C = \left(d'_{ij}\right)_{u_3\times v_3} + \left(c''_{ij}\right)_{u_3\times v_3} = \left(d'_{ij}+c''_{ij}\right)_{u_3\times v_3} = \left(f_{ij}\right)_{u_3\times v_3}$, where

$u_3 = \max\{u_1, r\} = \max\{\max\{m,p\}, r\} = \max\{m,p,r\} \geq u_1, u_2$,

$v_3 = \max\{v_1, s\} = \max\{\max\{n,q\}, s\} = \max\{n,q,s\} \geq v_1, v_2$ and

$d'_{ij} = \begin{cases} d_{ij}, & if\ 1\leq i\leq u_1,\ 1\leq j\leq v_1 \\ 0, & otherwise \end{cases}$, for $i = 1,2,\ldots, u_3$; $j = 1,2,\ldots, v_3$,

$c''_{ij} = \begin{cases} c_{ij}, & if\ 1\leq i\leq r,\ 1\leq j\leq s \\ 0, & otherwise \end{cases}$, for $i = 1,2,\ldots, u_3$; $j = 1,2,\ldots, v_3$ and

$f_{ij} = d'_{ij} + c''_{ij}$, for $i = 1,2,\ldots, u_3$; $j = 1,2,\ldots, v_3$. …………………… (3)

Now order of the matrix $A+(B+C)$ is $\max\{m,u_2\}\times\max\{n,v_2\}$ and $\max\{m,u_2\} = \max\{m,\max\{p,r\}\} = \max\{m,p,r\} = u_3$, $\max\{n,v_2\} = \max\{n,\max\{q,s\}\} = \max\{n,q,s\} = v_3$.

Therefore $A+(B+C)$ is a matrix of order $u_3\times v_3$.

Now, $+(B+C) = \left(a''_{ij}\right)_{u_3\times v_3} + \left(e'_{ij}\right)_{u_3\times v_3} = \left(a''_{ij}+e'_{ij}\right)_{u_3\times v_3} = \left(g_{ij}\right)_{u_3\times v_3}$, where

$a''_{ij} = \begin{cases} a_{ij}, & if\ 1\leq i\leq m,\ 1\leq j\leq n \\ 0, & otherwise \end{cases}$, for $i = 1,2,\ldots, u_3$; $j = 1,2,\ldots, v_3$,

$e'_{ij} = \begin{cases} e_{ij}, & if\ 1\leq i\leq u_2,\ 1\leq j\leq v_2 \\ 0, & otherwise \end{cases}$, for $i = 1,2,\ldots, u_3$; $j = 1,2,\ldots, v_3$ and

$g_{ij} = a''_{ij} + e'_{ij}$, for $i = 1,2,\ldots, u_3$; $j = 1,2,\ldots, v_3$. ………………….. (4)

Now, for $= 1,2,\ldots, u_3$; $j = 1,2,\ldots, v_3$,

$$f_{ij} = d'_{ij} + c''_{ij} = \begin{cases} d_{ij} + c''_{ij}, & for\ 1\leq i\leq u_1,\quad 1\leq j\leq v_1 \\ c''_{ij}, & otherwise \end{cases}$$

$$= \begin{cases} a'_{ij} + b'_{ij} + c''_{ij}, & for\ 1\leq i\leq u_1,\ 1\leq j\leq v_1 \\ c''_{ij}, & otherwise \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and for $i = 1,2,\ldots, u_3$; $j = 1,2,\ldots, v_3$,

$$g_{ij} = a''_{ij} + e'_{ij} = \begin{cases} a''_{ij} + e_{ij}, & for\ 1\leq i\leq u_2,\quad 1\leq j\leq v_2 \\ a''_{ij}, & otherwise \end{cases}$$

$$= \begin{cases} a''_{ij} + b''_{ij} + c'_{ij}, & for\ 1\leq i\leq u_2,\ 1\leq j\leq v_2 \\ a''_{ij}, & otherwise \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$\text{Now, } \left(a'_{ij}\right)_{u_1\times v_1} = \begin{pmatrix} A_{m\times n} & O_{m\times(v_1-n)} \\ O_{(u_1-m)\times n} & O_{(u_1-m)\times(v_1-n)} \end{pmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$\left(a''_{ij}\right)_{u_3\times v_3} = \begin{pmatrix} A_{m\times n} & O_{m\times(v_1-n)} & O_{m\times(v_3-v_1)} \\ O_{(u_1-m)\times n} & O_{(u_1-m)\times(v_1-n)} & O_{(u_1-m)\times(v_3-v_1)} \\ O_{(u_3-u_1)\times n} & O_{(u_3-u_1)\times(v_1-n)} & O_{(u_3-u_1)\times(v_3-v_1)} \end{pmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$\left(b'_{ij}\right)_{u_1\times v_1} = \begin{pmatrix} B_{p\times q} & O_{p\times(v_1-q)} \\ O_{(u_1-p)\times q} & O_{(u_1-p)\times(v_1-q)} \end{pmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$$\left(b''_{ij}\right)_{u_2\times v_2} = \begin{pmatrix} B_{p\times q} & O_{p\times(v_2-q)} \\ O_{(u_2-p)\times q} & O_{(u_2-p)\times(v_2-q)} \end{pmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

$$\left(c'_{ij}\right)_{u_2\times v_2} = \begin{pmatrix} C_{r\times s} & O_{r\times(v_2-s)} \\ O_{(u_2-r)\times s} & O_{(u_2-r)\times(v_2-s)} \end{pmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

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$$(c''_{ij})_{u_3\times v_3} = \begin{pmatrix} C_{r\times s} & O_{r\times(v_2-s)} & O_{r\times(v_3-v_2)} \\ O_{(u_2-r)\times s} & O_{(u_2-r)\times(v_2-s)} & O_{(u_2-r)\times(v_3-v_2)} \\ O_{(u_3-u_2)\times s} & O_{(u_3-u_2)\times(v_2-s)} & O_{(u_3-u_2)\times(v_3-v_2)} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots (12)$$

$$(d'_{ij})_{u_3\times v_3} = \begin{pmatrix} (d_{ij})_{u_1\times v_1} & O_{u_1\times(v_3-v_1)} \\ O_{(u_3-u_1)\times v_1} & O_{(u_3-u_1)\times(v_3-v_1)} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} A_{m\times n} & O_{m\times(v_1-n)} \\ O_{(u_1-m)\times n} & O_{(u_1-m)\times(v_1-n)} \end{pmatrix} + \begin{pmatrix} B_{p\times q} & O_{p\times(v_1-q)} \\ O_{(u_1-p)\times q} & O_{(u_1-p)\times(v_1-q)} \end{pmatrix} & O_{u_1\times(v_3-v_1)} \\ O_{(u_3-u_1)\times v_1} & O_{(u_3-u_1)\times(v_3-v_1)} \end{pmatrix} \ldots\ldots\ldots\ldots (13)$$

and , $(e'_{ij})_{u_3\times v_3} = \begin{pmatrix} (e_{ij})_{u_2\times v_2} & O_{u_2\times(v_3-v_2)} \\ O_{(u_3-u_2)\times v_2} & O_{(u_3-u_2)\times(v_3-v_2)} \end{pmatrix}$

$$= \begin{pmatrix} \begin{pmatrix} B_{p\times q} & O_{p\times(v_2-q)} \\ O_{(u_2-p)\times q} & O_{(u_2-p)\times(v_2-q)} \end{pmatrix} + \begin{pmatrix} C_{r\times s} & O_{r\times(v_2-s)} \\ O_{(u_2-r)\times s} & O_{(u_2-r)\times(v_2-s)} \end{pmatrix} & O_{u_2\times(v_3-v_2)} \\ O_{(u_3-u_2)\times v_2} & O_{(u_3-u_2)\times(v_3-v_2)} \end{pmatrix} \ldots\ldots\ldots (14)$$

Observing from (1) to (14) it is clear that for $1 \le i \le \min\{m,p,r\}$, $1 \le j \le \min\{n,q,s\}$, $f_{ij} = a_{ij} + b_{ij} + c_{ij} = g_{ij}$ and for other admissible values of i and j , f_{ij} = either $a_{ij} + b_{ij}$ or $b_{ij} + c_{ij}$ or $a_{ij} + c_{ij}$ or a_{ij} or b_{ij} or c_{ij} or 0 and g_{ij} = either $a_{ij} + b_{ij}$ or $b_{ij} + c_{ij}$ or $a_{ij} + c_{ij}$ or a_{ij} or b_{ij} or c_{ij} or 0 .

And, for these other admissible values of i and j , $f_{ij} = a_{ij} + b_{ij}$ iff $g_{ij} = a_{ij} + b_{ij}$, $f_{ij} = b_{ij} + c_{ij}$ iff $g_{ij} = b_{ij} + c_{ij}$,

$f_{ij} = a_{ij} + c_{ij}$ iff $g_{ij} = a_{ij} + c_{ij}$, etc.

Therefore 'addition' is associative.

We see that $O_{1\times1} = (0)_{1\times1} \in M(F)$ and $A + O_{1\times1} = (a_{ij})_{m\times n} + (0)_{1\times1} = (a_{ij})_{m\times n} = A$. (by definition(2.1)).

Hence $O_{1\times1}$ is the additive identity in $M(F)$. Hence the result.

Note (2.1) Clearly 'addition' on $M(F)$ is an extension of 'addition' on $M_{m\times n}(F)$, $\forall\, m,n \in \mathbb{N}$.

We see that $(M(F), +)$ is a commutative monoid with additive identity $O_{1\times1}$. Now for an $A = (a_{ij})_{m\times n} \in M(F)$ with $(m,n) \ne (1,1)$, there exists no matrix B in $M(F)$ such that $A + B = O_{1\times1}$. Hence $(M(F), +)$ is not a group.

Consider $M_1(F)$. Then $(M_1(F), +)$ is a commutative group with identity $O_{1\times1}$. Clearly 'addition' on $M(F)$ is an extension of the 'addition' on $M_1(F)$ and so $(M_1(F), +)$ is a subgroup of the monoid $(M(F), +)$ and we see that both have the same identity elements.

Consider $M_{m\times n}(F)$ with with $(m,n) \ne (1,1)$. We know that $(M_{m\times n}(F), +)$ is a commutative group. Since 'addition' on $M(F)$ is an extension of the 'addition' on $M_{m\times n}(F)$, so $(M_{m\times n}(F), +)$ is a subgroup of the monoid $(M(F), +)$. Now $O_{m\times n}$ is the identity in the group $(M_{m\times n}(F), +)$ and $O_{m\times n} \ne O_{1\times1}$.

Definition (2.2) For $m,n \in \mathbb{N}$, we define $I_{m\times n}$ as

$I_{m\times n} = (\delta_{ij})_{m\times n}$, where for $= 1,2,\ldots,m$; $j = 1,2,\ldots.,n$, $\delta_{ij} = \begin{cases} 1 , & if\ i = j \\ 0 , & if\ i \ne j \end{cases}$.

Definition (2.3) Define 'multiplication' of matrices in $M(F)$ by for all $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{p\times q} \in M(F)$,

$AB = (c_{ij})_{m\times q}$, where for $i = 1,2,\ldots.,m$; $j = 1,2,\ldots.,q$, $c_{ij} = \sum_{k=1}^{\min\{n,p\}} a_{ik}b_{kj}$.

Example (2.2) Consider the real matrices

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & -1 & 3 \end{pmatrix}_{3\times 4}, \quad B = \begin{pmatrix} 9 & 1 \\ 2 & 8 \\ 7 & 3 \\ 4 & 6 \\ -1 & 4 \end{pmatrix}_{5\times 2}.$$

Then $A + B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & -1 & 3 \end{pmatrix}_{3\times 4} + \begin{pmatrix} 9 & 1 \\ 2 & 8 \\ 7 & 3 \\ 4 & 6 \end{pmatrix}_{4\times 2} = \begin{pmatrix} 50 & 50 \\ 138 & 122 \\ 86 & 24 \end{pmatrix}_{3\times 2}$

Note (2.2) Clearly matrix multiplication is not commutative.

Theorem (2.2) $(M(F), \ .)$ is a non-commutative semi-group.

Proof : Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{p\times q}$, $C = (c_{ij})_{r\times s} \in M(F)$ be arbitrary.

Then $= (d_{ij})_{m\times q}$, $BC = (e_{ij})_{p\times s}$, where

for $= 1,2, \dots, m \ ; \ j = 1,2, \dots, q, \ d_{ij} = \sum_{k=1}^{\min\{n,p\}} a_{ik} b_{kj}$. ……………………… (1)

and for $i = 1,2, \dots, p \ ; \ j = 1,2, \dots, s, \ e_{ij} = \sum_{k=1}^{\min\{q,r\}} b_{ik} c_{kj}$. ……………………… (2)

Therefore $(AB)C = (f_{ij})_{m\times s}$ and $A(BC) = (g_{ij})_{m\times s}$, where

for $= 1,2, \dots, m \ ; \ j = 1,2, \dots, s, \ f_{ij} = \sum_{k=1}^{\min\{q,r\}} d_{ik} c_{kj}$. ……………………… (3)

and for $i = 1,2, \dots, m \ ; \ j = 1,2, \dots, s, \ g_{ij} = \sum_{k=1}^{\min\{n,p\}} a_{ik} e_{kj}$. ……………………… (4)

From (3), we get

for $i = 1,2, \dots, m \ ; \ j = 1,2, \dots, s, \ f_{ij} = \sum_{k=1}^{\min\{q,r\}} \left(d_{ij} = \sum_{t=1}^{\min\{n,p\}} a_{it} b_{tk}\right) c_{kj}$ (by (1))

$$= \sum_{t=1}^{\min\{n,p\}} a_{it} \left(\sum_{k=1}^{\min\{q,r\}} b_{tk} c_{kj}\right)$$

$$= \sum_{t=1}^{\min\{n,p\}} a_{it} e_{tj} \quad \text{(by (2))}$$

$$= g_{ij} \quad \text{(by (4)).}$$

Therefore 'multiplication' is associative.

From Note (2.2), it is clear that 'multiplication' is non-commutative .

Hence the result.

Note (2.3) Let $m, n \in \mathbb{N}$ and $A = (a_{ij})_{m\times n} \in M(F)$ be arbitrary. Let $I_{B(p,n)} = \begin{pmatrix} I_n \\ B_{p\times n} \end{pmatrix}$, where $B_{p\times n} \in M(F)$ is arbitrary and $p \in \mathbb{N}$ is arbitrary and let $I^{B(m,q)} = \begin{pmatrix} I_m & B_{m\times q} \end{pmatrix}$, where $B_{m\times q} \in M(F)$ is arbitrary and $q \in \mathbb{N}$ is arbitrary.

Then it is clear that $A.I_{B(p,n)} = A$ and $I^{B(m,q)}.A = A$; but $I_{B(p,n)}.A \neq A$ & $A.I^{B(m,q)} \neq A$, in general.

Again, $I_m.A = A = A.I_n$, But $I_n.A \neq A$ & $A.I_m \neq A$, if $m \neq n$.

If there exists $B \in M(F)$ such that $A.B = B.A = A$, then B must be a matrix of order $m \times n$ and the elements of B depends on the elements of A. Thus order of B depends on order of A and the elements of B depends on the elements of A.

Again it is obvious that $I_{m\times n} A_{m\times p} = A_{m\times p} \ iff \ n \geq m$ and $A_{p\times n} I_{m\times n} = A_{p\times n} \ iff \ m \geq n$; but

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$I_{m\times n}A_{m\times p} = \begin{pmatrix} A_{n\times p} \\ O_{(m-n)\times p} \end{pmatrix} \neq A_{m\times p}$, in general, and $I_n A_{m\times p} = A_{n\times p} \neq A_{m\times p}$ if $n < m$

Also, $A_{p\times n}I_{m\times n} = (A_{p\times m} \quad O_{p\times(n-m)}) \neq A_{p\times n}$, in general, and $A_{p\times n}I_m = A_{p\times m} \neq A_{p\times n}$ if $m < n$.

Now it can be easily proved that for given positive integers m, n, for all $A_{m\times n} \in M(F)$,

$A_{m\times n}I_{m\times n} = I_{m\times n}A_{m\times n} = A_{m\times n}$ iff $m = n$.

Hence there exists no matrix B in $M(F)$ such that $.C = C.B = C$, for all $C \in M(F)$. Hence $(M(F), .)$ is not a monoid.

Theorem (2.3) Matrix multiplication is distributive over matrix addition.

Proof : Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{p\times q}$, $C = (c_{ij})_{r\times s} \in M(F)$ be arbitrary.

Then $B + C = (b'_{ij})_{u\times v} + (c'_{ij})_{u\times v} = (b'_{ij} + c'_{ij})_{u\times v} = (d_{ij})_{u\times v}$, where $u = \max\{p, r\}$, $v = \max\{q, s\}$ and

$b'_{ij} = \begin{cases} b_{ij}\ , & if\ \ 1 \le i \le p\ ,\ 1 \le j \le q \\ 0\ , & otherwise \end{cases}$, for $i = 1,2, \dots, u$; $j = 1,2, \dots, v$ ………………………… (1)

$c'_{ij} = \begin{cases} c_{ij}\ , & if\ \ 1 \le i \le r\ ,\ 1 \le j \le s \\ 0\ , & otherwise \end{cases}$, for $i = 1,2, \dots, u$; $j = 1,2, \dots, v$ …………………………(2) and

$d_{ij} = b'_{ij} + c'_{ij}$, for $i = 1,2, \dots, u$; $j = 1,2, \dots, v$. ……………………………………………… (3)

Therefore $A.(B + C) = (e_{ij})_{m\times v}$, where

for $i = 1,2, \dots, m$; $j = 1,2, \dots, v$

$$e_{ij} = \sum_{k=1}^{\min\{n,u\}} a_{ik}d_{kj} = \sum_{k=1}^{\min\{n,u\}} a_{ik}(b'_{kj} + c'_{kj}) = \sum_{k=1}^{\min\{n,u\}} a_{ik}b'_{kj} + \sum_{k=1}^{\min\{n,u\}} a_{ik}c'_{kj} \dots \dots \dots (4)$$

Also $A.B = (f_{ij})_{m\times q}$, where for $i = 1,2, \dots, m$; $j = 1,2, \dots, q$

$$f_{ij} = \sum_{k=1}^{\min\{n,p\}} a_{ik}b_{kj} \quad \dots \dots \dots (5)$$

and $A.C = (g_{ij})_{m\times s}$, where for $i = 1,2, \dots, m$; $j = 1,2, \dots, s$

$$g_{ij} = \sum_{k=1}^{\min\{n,r\}} a_{ik}c_{kj} \quad \dots \dots \dots (6)$$

Therefore $(AB) + (AC) = (f_{ij})_{m\times q} + (g_{ij})_{m\times s} = (h_{ij})_{m\times v}$ (since $v = \max\{q, s\}$),

where for $i = 1,2, \dots, m$; $j = 1,2, \dots, v$, $h_{ij} = f'_{ij} + g'_{ij}$ …………………………………….. (7),

where for $i = 1,2, \dots, m$; $j = 1,2, \dots, v$, $f'_{ij} = \begin{cases} f_{ij}\ , & if\ \ 1 \le i \le m,\ 1 \le j \le q \\ 0, & otherwise \end{cases}$ and

$$g'_{ij} = \begin{cases} g_{ij}\ , & if\ \ 1 \le i \le m,\ 1 \le j \le s \\ 0, & otherwise \end{cases}.$$

i.e., for $i = 1,2, \dots, m; j = 1,2, \dots, v$, $f'_{ij} = \begin{cases} \sum_{k=1}^{\min\{n,p\}} a_{ik}b_{kj}\ , & if\ \ 1 \le i \le m,\ \ 1 \le j \le q \\ 0, & otherwise \end{cases}$ …………(8)

and $g'_{ij} = \begin{cases} \sum_{k=1}^{\min\{n,r\}} a_{ik}c_{kj}\ , & if\ \ 1 \le i \le m,\ \ 1 \le j \le s \\ 0, & otherwise \end{cases}$ …………………… (9)

(by (5) & (6))

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Therefore, for $i = 1,2, \dots, m$; $j = 1,2, \dots, v$,

$f'_{ij} = \sum_{k=1}^{\min\{n,u\}} a_{ik} b'_{kj}$ ……………………………… (10) (by (1) and (8), since $\min\{n,p\} \le \min\{n,u\}$)

and $g'_{ij} = \sum_{k=1}^{\min\{n,u\}} a_{ik} c'_{kj}$ ………………………….. (11) (by (2) and (9), since $\min\{n,r\} \le \min\{n,u\}$) .

From (4), (7), (10) and (11), it is clear that $(e_{ij})_{m\times v} = (h_{ij})_{m\times v}$,

$i.e.,\ A(B+C) = (AB) + (AC)$.

Therefore matrix multiplication is distributive over matrix addition from left.

Again $(B+C)A = (\alpha_{ij})_{u\times n}$, where for $i = 1,2, \dots, u$; $j = 1,2, \dots, n$,

$$\alpha_{ij} = \sum_{k=1}^{\min\{v,m\}} d_{ik} a_{kj} = \sum_{k=1}^{\min\{v,m\}} (b'_{ik} + c'_{ik}) a_{kj} = \sum_{k=1}^{\min\{v,m\}} b'_{ik} a_{kj} + \sum_{k=1}^{\min\{v,m\}} c'_{ik} a_{kj} \quad \dots\dots (12)$$

Also $BA = (\beta_{ij})_{p\times n}$, where for $i = 1,2, \dots, p$; $j = 1,2, \dots, n$,

$$\beta_{ij} = \sum_{k=1}^{\min\{q,m\}} b_{ik} a_{kj} \quad \dots\dots (13)$$

and $CA = (\gamma_{ij})_{r\times n}$, where for $i = 1,2, \dots, r$; $j = 1,2, \dots, n$,

$$\gamma_{ij} = \sum_{k=1}^{\min\{s,m\}} c_{ik} a_{kj} \quad \dots\dots (14)$$

Therefore $(BA) + (CA) = (\beta_{ij})_{p\times n} + (\gamma_{ij})_{r\times n} = (\theta_{ij})_{u\times n}$ (since, $u = \max\{p,r\}$),

where for $i = 1,2, \dots, u$; $j = 1,2, \dots, n$, $\theta_{ij} = \beta'_{ij} + \gamma'_{ij}$ ……………………………………… (15),

where for $i = 1,2, \dots, u$; $j = 1,2, \dots, n$, $\beta'_{ij} = \begin{cases} \beta_{ij} , & if\ 1 \le i \le p;\ 1 \le j \le n \\ 0, & otherwise \end{cases}$ and

$$\gamma'_{ij} = \begin{cases} \gamma_{ij} , & if\ 1 \le i \le r;\ 1 \le j \le n \\ 0, & otherwise \end{cases}$$

$i.e.$, for $i = 1,2, \dots, u$; $j = 1,2, \dots, n$, $\beta'_{ij} = \begin{cases} \sum_{k=1}^{\min\{q,m\}} b_{ik} a_{kj} , & if\ 1 \le i \le p;\ 1 \le j \le n \\ 0, & otherwise \end{cases}$ ………….(16)

and $\gamma'_{ij} = \begin{cases} \sum_{k=1}^{\min\{s,m\}} c_{ik} a_{kj} , & if\ 1 \le i \le r;\ 1 \le j \le n \\ 0, & otherwise \end{cases}$ ……………..(17)

(by (13) and (14))

Therefore, for $i = 1,2, \dots, u$; $j = 1,2, \dots, n$,

$\beta'_{ij} = \sum_{k=1}^{\min\{v,m\}} b'_{ik} a_{kj}$ …………………………....(18) (by (1) and (16), since $\min\{q,m\} \le \min\{v,m\}$)

and $\gamma'_{ij} = \sum_{k=1}^{\min\{v,m\}} c'_{ik} a_{kj}$ ………………………..(19) (by (2) and (17), since $\min\{s,m\} \le \min\{v,m\}$) .

Now from (12), (15), (18), (19), it is clear that $(\alpha_{ij})_{u\times n} = (\theta_{ij})_{u\times n}$,

$i.e., (B+C)A = (BA) + (CA)$.

Therefore matrix multiplication is distributive over matrix addition from right.

This completes the proof.

Definition (2.4) A matrix $A_{m\times n} \in M(F)$ is said to be non-zero if at least one element of $A_{m\times n}$ is non-zero in F .

Definition (2.5) Let R be a non-empty set on which two binary operations 'addition' and 'multiplication' are defined. Then the algebraic structure $(R, +, \ .)$ is said to be a weak hemi-ring if

(i) $(R, +)$ is a commutative monoid,

(ii) $(R, \ .)$ is a semi-group,

(iii) 'Multiplication' is distributive over 'addition',

but $a.0 = 0.a \neq 0$, in general, for $a \in R$, where 0 is the additive identity in R.

Note (2.4) From Theorem(2.1), Theorem(2.2), Theorem(2.3) and Definition(2.5) , we observed that $(M(F), +, \ .)$ is a weak hemi-ring with zero $O_{1\times 1}$.

Therefore from definition (2.4) , for all $m, n \in \mathbb{N}$, with $(m, n) \neq (1, 1)$, we see that $O_{m\times n} \in M(F)$ is neither a zero element nor a non-zero element in $M(F)$. But we call $O_{m\times n}$ as the (m, n)-zero matrix in $M(F)$.

Note (2.5) We know that $(M_{m\times n}(F), +)$ is a commutative group with additive identity $O_{m\times n}$. Again for all $A_{m\times n}, B_{m\times n} \in M_{m\times n}(F)$, the product $A_{m\times n}B_{m\times n} \in M_{m\times n}(F)$. Also, in extended matrix algebra, matrix multiplication is associative and matrix multiplication is distributive over matrix addition. Since these two properties are hereditary properties, we can say that

$(M_{m\times n}(F), +, \ .)$ is a ring with zero $O_{m\times n}$. Clearly this ring has no unity (as per Note (2.3)), provided $m \neq n$.

- **Extension of Linear Spaces of Linear Transformations :**

We shall establish that, if we consider a matrix as a linear transformation of vector spaces over a field F, then with respect to the extended addition (Definition(2.1)) of matrices, *i.e.*, linear transformations, and usual scalar multiplication of matrices, *i.e.*, linear transformations, the set $L(F)$ of all linear transformations from any vector space to any vector space, *i.,e.*, the set $M(F)$ forms a weak hemi-vector space (as per definition(2.8)) which is an extension of $L(V, W)$, for all vector spaces V, W over the field F.

Note (2.6) We know that any two vector spaces over the same field and of same dimension $n \in \mathbb{N}$ are isomorphic with respect to given ordered bases, and each is isomorphic to F^n.

Again, for $, n \in \mathbb{N}$, if $m \leq n$ then F^m can be thought as a subspace of F^n, since the map $f : F^m \longrightarrow F^n$, defined by $f(c_1, c_2, \dots\dots, c_m) = (c_1, c_2, \dots\dots, c_m, 0, 0, \dots\dots, 0), \forall (c_1, c_2, \dots\dots, c_m) \in F^m$

($n - m$ zeros are included)

is a mono-morphism from the vector space F^m to the vector space F^n over the field F.

Definition(2.6) (Addition on L(*F*)) Let $T, S \in L(F)$ be arbitrary. Then there exists vector spaces M, N, P, Q over the field F such that $T: N \rightarrow M$, $S: Q \rightarrow P$ are linear transformations. Let m, n, p, q be the dimensions of M, N, P, Q respectively.

Then for any given ordered bases $B_M = \{\alpha_1, \alpha_2, \dots, \alpha_m\}, B_N = \{\beta_1, \beta_2, \dots, \beta_n\}$, $B_P = \{\gamma_1, \gamma_2, \dots, \gamma_p\}$, $B_Q = \{\delta_1, \delta_2, \dots, \delta_q\}$ of M, N, P, Q respectively, T, S can be expressed as $T = (a_{ij})_{m\times n}$, $S = (b_{ij})_{p\times q}$ and $M \equiv F^m$, $N \equiv F^n$, $P \equiv F^p$, $Q \equiv F^q$.

Define $T + S$ as $T + S: F^s \rightarrow F^r$, where $r = \max\{m, p\}$, $s = \max\{n, q\}$, by

$$\forall\, \alpha = \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ \cdots \\ c_s \end{pmatrix}_{s\times 1} \in F^s, \ (T+S)(\alpha) = T(\alpha) + S(\alpha) = (a_{ij})_{m\times n}\begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ \cdots \\ c_s \end{pmatrix}_{s\times 1} + (b_{ij})_{p\times q}\begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ \cdots \\ c_s \end{pmatrix}_{s\times 1} \in F^r,$$

$i.e., T + S = (a_{ij})_{m\times n} + (b_{ij})_{p\times q}$.

Note(2.7) In definition(2.6), since $r = \max\{m, p\}$, $s = \max\{n, q\}$, it is clear that $r = m\ or\ p$ and $s = n\ or\ q$. It is obvious that, if $r = m, s = n$, then we consider the ordered bases B_M, B_N for co-domain and domain of $T + S$, respectively. For other cases, similarly we consider the suitable ordered bases.

Theorem(2.4) $(L(F), +)$ is a commutative monoid, but not a group.

Proof : From definition(2.6), it is clear that $L(F)$ is closed with respect to addition .

Let $T, S, R \in L(F)$ be arbitrary. Then there exists vector spaces M, N, P, Q, U, V over the field F such that $T: N \to M$, $S: Q \to P$, $R: V \to U$ are linear transformations. Let m, n, p, q, u, v be the dimensions of M, N, P, Q, U, V respectively.

Then for any given ordered bases $B_M = \{\alpha_1, \alpha_2, \ldots., \alpha_m\}, B_N = \{\beta_1, \beta_2, \ldots., \beta_n\}$, $B_P = \{\gamma_1, \gamma_2, \ldots., \gamma_p\}$, $B_Q = \{\delta_1, \delta_2, \ldots., \delta_q\}, B_U = \{\mu_1, \mu_2, \ldots., \mu_u\}, B_V = \{\tau_1, \tau_2, \ldots., \tau_v\}$ of M, N, P, Q, U, V respectively, T, S, R can be expressed as

$T = (a_{ij})_{m\times n}$, $S = (b_{ij})_{p\times q}$, $R = (c_{ij})_{u\times v} \in M(F)$.

Now $T + S = (a_{ij})_{m\times n} + (b_{ij})_{p\times q} = (b_{ij})_{p\times q} + (a_{ij})_{m\times n}$

(Since matrix addition is commutative in $M(F)$)

$= S + T$.

Therefore, ‘addition’ is commutative in $L(F)$.

Again, $(T + S) + R = \left((a_{ij})_{m\times n} + (b_{ij})_{p\times q}\right) + (c_{ij})_{u\times v} = (a_{ij})_{m\times n} + \left((b_{ij})_{p\times q} + (c_{ij})_{u\times v}\right)$

(Since matrix addition is associative in $M(F)$)

$= T + (S + R)$.

Therefore, ‘addition’ is associative in $L(F)$.

We see that $O \in L(F)$, where $O: F^1 \to F^1$ given by $O(t) = (0), \forall\, t \in F^1$.

Clearly, $O = O_{1\times 1} = (0)_{1\times 1}$. Now, $O + T = (0)_{1\times 1} + (a_{ij})_{m\times n} = (a_{ij})_{m\times n} = T$.

Therefore, O is identity in $L(F)$ with respect to addition.

Now for $m, n \in \mathbb{N}$ with $(m, n) \neq (1,1)$, and for any $T = (a_{ij})_{m\times n} \in L(F)$, it is clear that

$\forall\, S = (b_{ij})_{p\times q} \in L(F),\ T + S \neq\ O$.

Therefore $(L(F), +)$ is a commutative monoid, but not a group.

Definition(2.7)(Scalar Multiplication on $L(F)$) For all $T \in L(F)$, $c \in F$, if $T: V \to W$, where V, W are vector spaces over F, then the scalar multiplication of T by the scalar c is denoted by cT, is a mapping $cT: V \to W$, defined by $\forall\, \alpha \in V$, $(cT)(\alpha) = c(T(\alpha))$.

Note(2.8) From definition(2.7), for all $T \in L(F)$, $c \in F$, it is clear that, $cT \in L(F)$.

Theorem(2.5) Let F be a field. Then $\forall\, T, S \in L(F), \forall\, c, d \in F$,

(i) $c(T + S) = (cT) + (cS)$.

(ii) $(c + d)T = (cT) + (dT)$.

(iii) $c(dT) = (cd)T$.

(iv) $1.T = T$.

(v) $0.T = T.0 \neq O$, in general.

Proof : Trivial.

Definition(2.8) Let V be a commutative monoid with respect to a binary operation + (called addition of vectors, and elements of V are called vectors) having additive identity θ, called zero vector ; and F is a field such that there is an external operation $F \times V \rightarrow V$, called scalar multiplication. Then V is said to be a weak hemi-vector space over the field F if

(i) $c(\alpha + \beta) = (c\alpha) + (c\beta),\ \forall\, c \in F, \forall\, \alpha, \beta \in V$,

(ii) $(c + d)\alpha = (c\alpha) + (d\alpha), \forall\, c, d \in F, \forall\, \alpha \in V$,

(iii) $c(d\alpha) = (cd)\alpha, \forall\, c, d \in F, \forall\, \alpha \in V$,

(iv) $1.\alpha = \alpha, \forall\, \alpha \in V$,

But $0.\alpha$ $(= \alpha.0)$ may not be equal to θ, $\forall \alpha \in V$.

Note(2.9) From Theorem(2.4), Theorem(2.5) and definition(2.8), we can say that $L(F)$ is a weak hemi-vector space over the field F. Also it is clear that this weak hemi-vector space is an extension of all the linear spaces of linear transformations over the field F.

➢ **Some Properties of the weak Hemi-ring $(\boldsymbol{M(F)}, +, \ .)$**

Definition (2.9) For a given nonzero matrix $A_{m\times n} \in M(F)$, if there exists a matrix $B_{p\times q} \in M(F)$

such that $A_{m\times n}B_{p\times q} = I_{m\times q}$ then $B_{p\times q}$ is called a right inverse of $A_{m\times n}$ and if $B_{p\times q}A_{m\times n} = I_{p\times n}$, then $B_{p\times q}$ is called a left inverse of $A_{m\times n}$.

Theorem (2.6) **Le**t $m \leq n$. Then for two non-zero matrices $A_{m\times n}$, $B_{m\times n} \in M(F)$,

$A_{m\times n}B_{m\times n} = I_{m\times n}$ iff $A_{m\times m}B_{m\times m} = B_{m\times m}A_{m\times m} = I_m$ and for $j = m+1, m+2, \ldots, n$, each

j^{th} column B_j (say) of $B_{m\times n}$ is zero.

Proof : Let $A_{m\times n} = (a_{ij})_{m\times n}$, $B_{m\times n} = (b_{ij})_{m\times n}$.

Let $A_{m\times n}B_{m\times n} = I_{m\times n}$ ………………………………………. (1)

Then from (1), we have $A_{m\times n}B_{m\times n} = \begin{pmatrix} I_m & , & O_{m\times(n-m)} \end{pmatrix}$ ………………………… (2)

Since $\leq n$, from (2), we get $A_{m\times m}B_{m\times m} = I_m$ ……………..………………. (3)

From (3), it is clear that $B_{m\times m}$ is invertible , and hence $B_{m\times m}A_{m\times m} = I_m$ …………… (4)

From (3) and (4), we get $A_{m\times m}B_{m\times m} = B_{m\times m}A_{m\times m} = I_m$.

Now $m \leq n$ implies that, $A_{m\times n}B_{m\times n} = A_{m\times m}B_{m\times n}$

$$= (A_{m\times m}B_1\, ,\ A_{m\times m}B_2\, ,\ \ldots\, ,\ A_{m\times m}B_m\, ,\ A_{m\times m}B_{m+1}\, ,\ \ldots\, ,\ A_{m\times m}B_n)$$

$$= I_{m\times n} \ \text{(given)}$$

$$= \begin{pmatrix} I_m\, , & O_{m\times(n-m)} \end{pmatrix} .$$

Therefore, $A_{m\times m}B_j = O_{m\times 1}$, for $j = m+1, m+2, \ldots, n$.

This implies that $B_j = O_{m\times 1}$, for $j = m+1, m+2, \ldots, n$. (since $A_{m\times m}$ is invertible).

Conversely, let $A_{m\times m}B_{m\times m} = B_{m\times m}A_{m\times m} = I_m$ …………………………….. (5)

and $B_j = O_{m\times 1}$, for $j = m+1, m+2, \ldots, n$ ……………………………. (6)

Now $m \le n$ implies that, $A_{m\times n}B_{m\times n} = A_{m\times m}B_{m\times n}$

$$= (A_{m\times m}B_{m\times m} ,\ A_{m\times m}B_{m+1} ,\ \ldots ,\ A_{m\times m}B_n)$$

$$= (I_m ,\ O_{m\times 1} ,\ \ldots ,\ O_{m\times 1})_{m\times n} \quad \text{(by (5) and (6))}$$

$$= I_{m\times n} .$$

This completes the proof.

Corollary (2.6.a) Let $n \le m$. Then for two non-zero matrices $A_{m\times n} , B_{m\times n} \in M(F)$,

$A_{m\times n}B_{m\times n} = I_{m\times n}$ iff $A_{n\times n}B_{n\times n} = B_{n\times n}A_{n\times n} = I_n$ and for $i = n+1, n+2, \ldots, m$, each i^{th} row A_i (say) of $A_{m\times n}$ is zero.

Proof: Taking transposition both sides of $A_{m\times n}B_{m\times n} = I_{m\times n}$ and following the same procedure, as in Theorem (2.6), we have the result .

Corollary (2.6.b) Let $m \le n$. Then for two non-zero matrices $A_{m\times n} , B_{m\times n} \in M(F)$,

$A_{m\times n}B_{m\times n} = B_{m\times n}A_{m\times n} = I_{m\times n}$ iff $A_{m\times m}B_{m\times m} = B_{m\times m}A_{m\times m} = I_m$ and

for $j = m+1, m+2, \ldots, n$, each j^{th} column of $A_{m\times n}$ and $B_{m\times n}$ are zero.

Proof : Immediately follows from Theorem (2.6) .

Corollary (2.6.c) Let $n \le m$. Then for two non-zero matrices $A_{m\times n} , B_{m\times n} \in M(F)$,

$A_{m\times n}B_{m\times n} = B_{m\times n}A_{m\times n} = I_{m\times n}$ iff $A_{n\times n}B_{n\times n} = B_{n\times n}A_{n\times n} = I_n$ and

for $i = n+1, n+2, \ldots, m$, each i^{th} row of $A_{m\times n}$ and $B_{m\times n}$ are zero.

Proof : Taking transposition in $A_{n\times n}B_{n\times n} = B_{n\times n}A_{n\times n} = I_n$, immediately follows from

Theorem (2.6) .

Theorem (2.7) Let $A_{m\times n} , B_{p\times q} \in M(F)$. Then $A_{m\times n}B_{p\times q} = I_{m\times q}$ iff

(i) $A_{m\times n}B_{n\times m} = I_m$ and $A_{m\times n}\big(B_{m+1}, B_{m+2} , \ldots, B_q\big)_{p\times(q-m)} = O_{m\times(q-m)}$, if $n \le p$, $m < q$; where

for $= m+1, m+2, \ldots, q$; B_j is the j^{th} column of $B_{p\times q}$.

(ii) $A_{q\times n}B_{n\times q} = I_q$ and $\begin{pmatrix} R_{q+1} \\ R_{q+2} \\ \ldots\ldots \\ \ldots\ldots \\ R_m \end{pmatrix}_{(m-q)\times n} B_{p\times q} = O_{(m-q)\times q}$, if $n \le p$, $m > q$; where for

$i = q+1, q+2, \ldots, m$; R_i is the i^{th} row of $A_{m\times n}$.

(iii) $A_{m\times p}B_{p\times m} = I_m$ and $A_{m\times n}\big(B_{m+1}, B_{m+2} , \ldots , B_q\big)_{p\times(q-m)} = O_{m\times(q-m)}$, if $n > p$, $m < q$.

(iv) $A_{q\times p}B_{p\times q} = I_q$ and $\begin{pmatrix} R_{q+1} \\ R_{q+2} \\ \ldots\ldots \\ \ldots\ldots \\ R_m \end{pmatrix}_{(m-q)\times n} B_{p\times q} = O_{(m-q)\times q}$, if $n > p$, $m > q$.

Proof : (i) Let $n \le p$, $m < q$.

Firstly, let $A_{m\times n}B_{p\times q} = I_{m\times q}$ ………………….. (1)

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Then (1) implies that $A_{m\times n}B_{n\times q} = I_{m\times q}$ (since $n \le p$, hence $A_{m\times n}B_{p\times q} = A_{m\times n}B_{n\times q}$).

i.e., $A_{m\times n}\left(B_{n\times m}\,,\ B_{m+1}\,,\ B_{m+2}\,,\ldots\ldots\,,\ B_q\right) = \left(I_m\ ,\ \ O_{m\times(q-m)}\right)$ (since $m < q$).

i.e., $A_{m\times n}B_{n\times m} = I_m$ and $A_{m\times n}\left(B_{m+1}\,,\ B_{m+2}\,,\ldots\ldots\,,\ B_q\right)_{p\times(q-m)} = O_{m\times(q-m)}$.

Conversely, let $A_{m\times n}B_{n\times m} = I_m$ and $A_{m\times n}\left(B_{m+1}\,,\ B_{m+2}\,,\ldots\ldots\,,\ B_q\right)_{p\times(q-m)} = O_{m\times(q-m)}$.

Then $A_{m\times n}\left(B_{n\times m}\,,\ B_{m+1}\,,\ B_{m+2}\,,\ldots\ldots\,,\ B_q\right) = \left(I_m\ ,\ \ O_{m\times(q-m)}\right)$

i.e. , $A_{m\times n}B_{n\times q} = I_{m\times q}$. This implies that $A_{m\times n}B_{p\times q} = I_{m\times q}$ (since $n \le p$).

Hence the result.

Similarly we can prove (ii), (iii) and (iv) .

Theorem (2.8) For a given nonzero matrix $A_{m\times n}$ in $M(F)$, if there exists $B_{p\times q}$ in $M(F)$ such that $A_{m\times n}B_{p\times q} = I_{m\times q}$, then

$m \le n$, except the case $m > n > q$.

Proof : Let $A_{m\times n} = \left(a_{ij}\right)_{m\times n}$ and $B_{p\times q} = \left(b_{ij}\right)_{p\times q}$. Now $A_{m\times n}B_{p\times q} = I_{m\times q}$ gives

$\sum_{k=1}^{\min\{n,p\}} a_{ik}b_{kj} = \delta_{ij}\ ,\ for\ i = 1,2,\ldots\ldots,m\,;\ j = 1,2,\ldots\ldots,q$ ……………..…………….. (1)

From (1) it is clear that if $n > p$ then all the entries from (p + 1)-th column to the n-th column of $A_{m\times n}$ are not present in (1) and if $p > n$ then all the entries from (n + 1)-th row to the p-th row of $B_{p\times q}$ are not present in (1).

Therefore to establish the result it is sufficient to prove the proposition when $n = p$. In this case, (1) becomes

$\sum_{k=1}^{n} a_{ik}b_{kj} = \delta_{ij}\ ,\ for\ i = 1,2,\ldots\ldots,m\,;\ j = 1,2,\ldots\ldots,q$ ……………………………….….. (2)

If possible, let $m > n$. Then from (2), we can say that rank of $A_{m\times n}$ is less than or equal to $n - 1$,

if $q \ge m > n$ or $m > q \ge n$ and the rank of $A_{m\times n}$ is q , if $m > n > q$.

Case (I) : Let $q \ge m > n$. Since rank of $A_{m\times n}$ is less than or equal to $n - 1$, the system of linear equations (2) for the unknown $b_{ij}{}'s$ will be equivalent to at least

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \vdots & a_{(n-1)n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times n} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nq} \end{pmatrix}_{n\times q}$$

m-th

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \vdots & 0 & 0 & \vdots & 0 & 0 & \vdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \vdots & 1 & 0 & \vdots & 0 & 0 & \vdots & 0 \\ c_{n1} & c_{n2} & \vdots & c_{n(n-1)} & 1 & \vdots & 0 & 0 & \vdots & 0 \\ c_{(n+1)1} & c_{(n+1)2} & \vdots & c_{(n+1)(n-1)} & c_{(n+1)n} & \vdots & 0 & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{m(n-1)} & c_{mn} & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}_{m\times q}$$

which is not possible, since the $(m, m) - entry$ in the Left Hand Side is zero but that in the Right Hand Side is 1.

Case (II) : Let $m > q \geq n$. Since rank of $A_{m\times n}$ is less than or equal to $n-1$, the system of linear equations (2) for the unknown $b_{ij}'s$ will be equivalent to at least

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times n} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nq} \end{pmatrix}_{n\times q}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & 1 & 0 & \vdots & 0 \\ c_{n1} & c_{n2} & \vdots & c_{n(n-1)} & 1 & \vdots & 0 \\ c_{(n+1)1} & c_{(n+1)2} & \vdots & c_{(n+1)(n-1)} & c_{(n+1)n} & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{q1} & c_{q2} & \vdots & c_{q(n-1)} & c_{qn} & \vdots & 1 \\ c_{(q+1)1} & c_{(q+1)2} & \vdots & c_{(q+1)(n-1)} & c_{(q+1)n} & \vdots & c_{(q+1)q} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ c_{m1} & c_{m2} & & c_{m(n-1)} & c_{mn} & & c_{mq} \end{pmatrix}_{m\times q}$$

which is not possible, since the $(n, n) - entry$ in the Left Hand Side is zero but that in the Right Hand Side is 1.

Case (III) : Let $m > n > q$. Since rank of $A_{m\times n}$ is equal to q, the system of linear equations (2) for the unknown $b_{ij}'s$ will be equivalent to

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(q-1)} & a_{1q} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(q-1)} & a_{2q} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ a_{q1} & a_{q2} & \vdots & a_{q(q-1)} & a_{qq} & \cdots & a_{qn} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & & 0 & \cdots & 0 \end{pmatrix}_{m\times n} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nq} \end{pmatrix}_{n\times q}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & 1 \\ c_{(q+1)1} & c_{(q+1)2} & \vdots & c_{(q+1)q} \\ c_{(q+2)1} & c_{(q+2)2} & \vdots & c_{(q+2)q} \\ \vdots & \vdots & \cdots & \vdots \\ c_{m1} & c_{m2} & & c_{mq} \end{pmatrix}_{m\times q}$$

which may happen, provided all c_{ij} of the Right Hand Side are zero.

This completes the proof.

Theorem (2.9) For a given nonzero matrix $B_{p\times q}$ in $M(F)$, if there exists $A_{m\times n}$ in $M(F)$ such that

$A_{m\times n}B_{p\times q} = I_{m\times q}$ then $q \leq n$, except the case $q > n > m$.

Proof : In the proof of Theorem(2.8) taking transpose both sides of $A_{m\times n}B_{p\times q} = I_{m\times q}$ and interchanging the matrices A and B we shall get the result.

3. CONCLUSION

Further study may be continued to observe different properties of traditional matrix algebra in the extended matrix algebra.

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